

Available online at [www.sciencedirect.com](http://www.sciencedirect.com)**ScienceDirect**

Procedia Engineering 79 (2014) 179 – 183

**Procedia  
Engineering**[www.elsevier.com/locate/procedia](http://www.elsevier.com/locate/procedia)

37th National Conference on Theoretical and Applied Mechanics (37th NCTAM 2013) &amp; The 1st International Conference on Mechanics (1st ICM)

## Laminated tubes under extension

Wei-Der Tseng<sup>a</sup>, Jiann-Quo Tarn<sup>b</sup><sup>a</sup>Department of Construction Engineering, NanJen University of Science and Technology, Tainan 73746, Taiwan, ROC<sup>b</sup>Department of Civil Engineering, National Cheng Kung University, Tainan 70101, Taiwan, ROC

### Abstract

An exact analysis of deformation and stress field in a laminated elastic tube under extension is presented. The problem is formulated on the basis of the state space formalism for axisymmetric deformation of transversely isotropic layer. The transfer matrix transmits the state vector in radial direction from inner surface to outer surface and takes into account the interfacial continuity and lateral boundary conditions in a regular manner. Upon delineating the symplectic orthogonality relations of the eigenvectors and by using eigenfunction expansion, a rigorous solution which satisfies the end conditions is determined.

© 2014 Elsevier Ltd. This is an open access article under the CC BY-NC-ND license

(<http://creativecommons.org/licenses/by-nc-nd/3.0/>).

Selection and peer-review under responsibility of the National Tsing Hua University, Department of Power Mechanical Engineering

**Keywords:** state space approach; laminated tube; transfer matrix.

### Nomenclature

|                      |   |
|----------------------|---|
| $r, \theta, z$       | cylindrical coordinates   |
| $u$                  | displacement  |
| $c_{ij}$             | elastic constants with reference to the cylindrical coordinates |
| <i>Greek symbols</i> |   |
| $\sigma$             | stress  |
| $\mu$                | eigenvalue  |
| $\Psi$               | eigensolution   |
| <i>Subscripts</i>    |   |
| 1, 2, 3              | stand for $r, \theta, z$  |

\* Corresponding author. Tel.: +0-007-522-8790; fax: +0-006-653-3854.

E-mail address: [wdseng@mail.nju.edu.tw](mailto:wdseng@mail.nju.edu.tw)

## 1. Introduction

In this paper we present an exact analysis of the displacement and stress fields in a laminated tube of finite length under extension. Since the loads are axially symmetric and the materials are transversely isotropic, the equations of elasticity for axisymmetric deformation are formulated into a state equation and an output equation of laminated layer. To determine a rigorous solution of the problem, it is necessary to find a complete solution of the state equation and make it satisfy the prescribed BC. The transfer matrix transmits the state vector in radial direction from inner surface to outer surface and takes into account the interfacial continuity and lateral boundary conditions in a regular manner. Regardless the number of layers, its determination requires only eigensolutions of matrices. We also delineate the symplectic orthogonality relations of the eigenvectors, and by means of eigenfunction expansion, an exact analysis of the displacement and stress fields in the laminated tube with the bottom plane perfectly bonded is carried out.

## 2. State space formulation

Consider a transversely isotropic circular tube of laminated cross-section resting on a rigid base. Let the origin of the cylindrical coordinates  $(r, \theta, z)$  be located at the center of the bottom plane, with the  $z$  axis pointing upward. When the tube is subjected to axisymmetric loadings, the deformation and stress fields are independent of  $\theta$ . On the basis of the state space formalism for anisotropic elasticity [1], the state equation and output equation [2] of the  $k$ -th layer are as follows:

$$\frac{\partial}{\partial z} \begin{bmatrix} u_r \\ u_z \\ r\sigma_{rz} \\ r\sigma_{zz} \end{bmatrix}_k = \begin{bmatrix} 0 & -\partial_r & r^{-1}c_{44}^{-1} & 0 \\ l_{31} & 0 & 0 & r^{-1}c_{33}^{-1} \\ l_{41} & 0 & 0 & l_{46} \\ 0 & 0 & -\partial_r & 0 \end{bmatrix} \begin{bmatrix} u_r \\ u_z \\ r\sigma_{rz} \\ r\sigma_{zz} \end{bmatrix}_k, \quad \begin{bmatrix} r\sigma_{rr} \\ r\sigma_{\theta\theta} \end{bmatrix}_k = \begin{bmatrix} \tilde{c}_{11}(r\partial_r + 1) - 2c_{66} & \hat{c}_{13} \\ \tilde{c}_{12}(r\partial_r + 1) + 2c_{66} & \hat{c}_{13} \end{bmatrix}_k \begin{bmatrix} u_r \\ r\sigma_{zz} \end{bmatrix}_k, \quad (1)$$

where it is understood that the indices 1, 2, 3 stand for  $r, \theta$  and  $z$ , respectively. With  $c_{66} = (c_{11} - c_{12})/2$ , there are five independent elastic constants for transversely isotropic materials.  $\partial_r, \partial_z$  denote partial differentiation with respect to  $r$  and  $z$ , respectively, and

$$l_{31} = -\hat{c}_{13}(\partial_r + r^{-1}), \quad l_{41} = -\tilde{c}_{11}[\partial_r(r\partial_r) - r^{-1}], \quad l_{46} = -\hat{c}_{13}(\partial_r - r^{-1}), \quad \tilde{c}_{ij} = c_{ij} - c_{i3}c_{33}^{-1}c_{j3}, \quad \hat{c}_{ij} = c_{ij}c_{33}^{-1}.$$

The cylindrical surfaces at  $r = a$  and  $r = b$  are traction-free:  $\begin{bmatrix} \sigma_{rr} & \sigma_{rz} \end{bmatrix}_{r=a} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \sigma_{rr} & \sigma_{rz} \end{bmatrix}_{r=b} = \begin{bmatrix} 0 & 0 \end{bmatrix}$

The bottom plane is assumed to be perfectly bonded with the rigid base (the fixed end), thus the end conditions at the bottom plane  $z = 0$  and the upper plane at  $z = l$  are  $\begin{bmatrix} u_r & u_z \end{bmatrix}_{z=0} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \sigma_{rz} & \sigma_{zz} \end{bmatrix}_{z=l} = \begin{bmatrix} 0 & p(r) \end{bmatrix}$

## 3. Eigensolution

We seek the solution of Eq. (1) of the form

$\begin{bmatrix} u_r & u_z & r\sigma_{rz} & r\sigma_{zz} \end{bmatrix}_k = \begin{bmatrix} c_1 J_1(\lambda r) & c_2 J_0(\lambda r) & c_3 r J_1(\lambda r) & c_4 r J_0(\lambda r) \end{bmatrix}_k e^{\mu z}$ , where  $\mu$  is a parameter to be determined;  $J_0(\lambda r)$  and  $J_1(\lambda r)$  are the Bessel functions of the first kind [3], of order 0 and 1, respectively;  $\lambda$  and  $C_i$  ( $i = 1, 2, 3, 4$ ) are constants to be determined. Substituting it into Eq. (1), we arrive at

$$\begin{bmatrix} 0 & \lambda & c_{44}^{-1} & 0 \\ -\hat{c}_{13}\lambda & 0 & 0 & c_{33}^{-1} \\ \tilde{c}_{11}\lambda^2 & 0 & 0 & \hat{c}_{13}\lambda \\ 0 & 0 & -\lambda & 0 \end{bmatrix}_k \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}_k = \mu \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}_k, \quad (2)$$

to which non-trivial solutions exist if and only if the determinant of the coefficient matrix equals to zero. This condition yields  $\{c_{11}c_{44}\kappa^4 - [c_{11}c_{33} - c_{13}(c_{13} + 2c_{44})]\kappa^2 + c_{33}c_{44}\}_k = 0$ ,  $(\kappa_k = \lambda_k / \mu)$  which has four roots. To each  $\kappa_{jk}$  there corresponds a solution of Eq. (2) determined within a constant  $A_{jk}$ .

It can be shown that there are two linearly independent solutions, and the solution of Eq. (1) is

$$[u_r \quad u_z \quad r\sigma_{rz} \quad r\sigma_{zz} \quad r\sigma_{rr} \quad r\sigma_{\theta\theta}]_k = \sum_{j=1}^2 A_{jk} e^{\mu r} [f_1(\kappa_j, \mu, r) \quad f_2(\kappa_j, \mu, r) \quad f_3(\kappa_j, \mu, r) \quad f_4(\kappa_j, \mu, r) \quad f_5(\kappa_j, \mu, r) \quad f_6(\kappa_j, \mu, r)]_k, \quad (3)$$

$$\begin{aligned} \text{where } & [f_1(\kappa_j, \mu, r) \quad f_2(\kappa_j, \mu, r) \quad f_3(\kappa_j, \mu, r) \quad f_4(\kappa_j, \mu, r)]_k \\ & = [\kappa_j^2 c_{44} - c_{33} J_1(\lambda r) \quad \kappa_j (c_{44} + c_{13}) J_0(\lambda r) \quad -\mu c_{44} (c_{33} + \kappa_j^2 c_{44}) r J_1(\lambda r) \quad \kappa_j \mu c_{44} (\kappa_j^2 c_{13} + c_{33}) r J_0(\lambda r)]_k, \\ [f_5(\kappa_j, \mu, r)]_k & = \{[c_{44}(\kappa_j^2 c_{11} + c_{13}) - \tilde{c}_{11} c_{33}] \kappa_j \mu r J_0(\kappa_j \mu r) - 2c_{66}(\kappa_j^2 c_{44} - c_{33}) J_1(\kappa_j \mu r)\}_k, \\ [f_6(\kappa_j, \mu, r)]_k & = \{[c_{44}(\kappa_j^2 c_{12} + c_{13}) - \tilde{c}_{12} c_{33}] \kappa_j \mu r J_0(\kappa_j \mu r) + 2c_{66}(\kappa_j^2 c_{44} - c_{33}) J_1(\kappa_j \mu r)\}_k. \end{aligned}$$

It can also be shown that there are another two linearly independent solutions derivable from  $[u_r \quad u_z \quad r\sigma_{rz} \quad r\sigma_{zz}]_k = [c_1 Y_1(\lambda r) \quad c_2 Y_0(\lambda r) \quad c_3 r Y_1(\lambda r) \quad c_4 r Y_0(\lambda r)]_k e^{\mu r}$ , and we have

$$[u_r \quad u_z \quad r\sigma_{rz} \quad r\sigma_{zz} \quad r\sigma_{rr} \quad r\sigma_{\theta\theta}]_k = \sum_{j=1}^2 B_{jk} e^{\mu r} [g_1(\kappa_j, \mu, r) \quad g_2(\kappa_j, \mu, r) \quad g_3(\kappa_j, \mu, r) \quad g_4(\kappa_j, \mu, r) \quad g_5(\kappa_j, \mu, r) \quad g_6(\kappa_j, \mu, r)]_k, \quad (4)$$

where  $g_i(\kappa_j, \mu, r)$  ( $i = 1, 2, \dots, 6$ ) are similar to  $f_i(\kappa_j, \mu, r)$ .

#### 4. Satisfaction of the boundary conditions at $r = a$ and $r = b$

From Eqs. (3) and (4) the displacements and stresses of the  $k$ -th layer in radial direction can be written as  $[u_r \quad u_z \quad r\sigma_{rr} \quad r\sigma_{rz}]_k = [U_r(r) \quad U_z(r) \quad S_{rr}(r) \quad S_{rz}(r)]_k e^{\mu r}$ , where

$$\begin{bmatrix} U_r(r) \\ U_z(r) \\ S_{rr}(r) \\ S_{rz}(r) \end{bmatrix}_k = \begin{bmatrix} f_1(\kappa_1, \mu, r) & f_1(\kappa_2, \mu, r) & g_1(\kappa_1, \mu, r) & g_1(\kappa_2, \mu, r) \\ f_2(\kappa_1, \mu, r) & f_2(\kappa_2, \mu, r) & g_2(\kappa_1, \mu, r) & g_2(\kappa_2, \mu, r) \\ f_5(\kappa_1, \mu, r) & f_5(\kappa_2, \mu, r) & g_5(\kappa_1, \mu, r) & g_5(\kappa_2, \mu, r) \\ f_6(\kappa_1, \mu, r) & f_6(\kappa_2, \mu, r) & g_6(\kappa_1, \mu, r) & g_6(\kappa_2, \mu, r) \end{bmatrix}_k \begin{bmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{bmatrix}_k. \quad (5)$$

Substituting the internal radii  $r = c_{k-1}$  of the  $k$ -th layer to Eq. (5), we obtain  $[A_1 \quad A_2 \quad B_1 \quad B_2]_k$  in terms of  $[U_r(c_{k-1}) \quad U_z(c_{k-1}) \quad S_{rr}(c_{k-1}) \quad S_{rz}(c_{k-1})]_k$ . Then substituting back to (5), the solution can be rewritten in the form:  $\mathbf{R}_k(r) = \mathbf{P}_k(r; c_{k-1}) \mathbf{R}_k(c_{k-1})$ , where the matrices and vectors are defined as

$$\begin{aligned} \mathbf{R}_k(r) &= [U_r(r) \quad U_z(r) \quad S_{rr}(r) \quad S_{rz}(r)]_k^T, \quad \mathbf{P}_k(r; c_{k-1}) = \mathbf{M}_k(r) \mathbf{M}_k^{-1}(c_{k-1}), \\ \mathbf{M}_k(r) &= [f_1(\kappa_1, \mu, r) \quad f_2(\kappa_2, \mu, r) \quad g_1(\kappa_1, \mu, r) \quad g_2(\kappa_2, \mu, r)]_k^T, \quad \mathbf{f}_j = [f_1(\kappa_j, \mu, r) \quad f_2(\kappa_j, \mu, r) \quad f_5(\kappa_j, \mu, r) \quad f_6(\kappa_j, \mu, r)]_k^T, \\ \mathbf{g}_j &= [g_1(\kappa_j, \mu, r) \quad g_2(\kappa_j, \mu, r) \quad g_5(\kappa_j, \mu, r) \quad g_6(\kappa_j, \mu, r)]_k^T. \end{aligned}$$

Specifically, the interfacial continuity conditions at external radii  $r = c_k$  of  $k$ -th layer is  $\mathbf{R}_{k+1}(c_k) = \mathbf{R}_k(c_k)$ . It follows that  $\mathbf{R}_{k+1}(r) = \mathbf{P}_k(r; c_{k-1})\mathbf{R}_k(c_{k-1})$ , which transfers the state vector from the  $k$ -th layer to the  $(k+1)$ -th layer. Transferring the vector recursively from the inner radii  $r = a$  outward yields  $\mathbf{R}_k(r) = \mathbf{T}_k(r)\mathbf{R}_1(a)$ , where the transfer matrix:  $\mathbf{T}_k(r) = \begin{cases} \mathbf{P}_1(r; a), & k=1; \\ \mathbf{P}_k(r; c_{k-1})\mathbf{T}_{k-1}(c_{k-1}), & k=2, 3, \dots, m. \end{cases}$  Following the same line by using the transfer matrix, the eigenvalue  $\mu_j$  ( $j = 1, 2, \dots$ ) and the associated eigenvector are determined by satisfying the interfacial continuity conditions and the traction-free conditions on the inner and outer surfaces. Setting the outer radii  $r = b$  gives  $\mathbf{R}_m(b) = \mathbf{T}_m(b)\mathbf{R}_1(a)$ , which can be expressed as

$$\begin{bmatrix} \mathbf{U}_m(b) \\ \mathbf{S}_m(b) \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{uu}(b) & \mathbf{T}_{us}(b) \\ \mathbf{T}_{su}(b) & \mathbf{T}_{ss}(b) \end{bmatrix}_m \begin{bmatrix} \mathbf{U}_1(a) \\ \mathbf{S}_1(a) \end{bmatrix}. \quad (6)$$

The traction-free BC on  $r = a$  and  $b$  demand  $\mathbf{S}_1(a) = 0$ ,  $\mathbf{S}_m(b) = 0$ . Imposing them on Eq. (6) yields  $[\mathbf{T}_{su}(b)]_m = 0$ , from which the eigenvalues  $\mu$  are determined. Linearly independent eigenvectors associated with  $\mu_0 = 0$  can be determined by considering the Jordan-chain solution and are given as

$$\boldsymbol{\Psi}_0^{(0k)} = [0 \quad 1 \quad 0 \quad 0]^T, \quad \boldsymbol{\Psi}_0^{(1k)} = [D_1^{(k)}r + D_1'^{(k)}r^{-1} \quad 0 \quad 0 \quad 2(c_{13})_{(k)}D_1^{(k)}r + (c_{33})_{(k)}r]^T; \text{ the constants } D_1^{(k)}, D_1'^{(k)} \text{ can be expressed by transfer matrices and } D_1^{(1)} \text{ can be determined.}$$

A linear combination of the eigensolutions produces the complete solution of the  $k$ -th layer:

$$\begin{bmatrix} u_r & u_z & r\sigma_{rz} & r\sigma_{zz} \end{bmatrix}_k^T = (C_0\boldsymbol{\Psi}_0^{(0k)} + C'_0\boldsymbol{\Psi}_0^{(1k)}) + \sum_{i=1}^{\infty} [C_i\boldsymbol{\Psi}_i^{(k)}e^{\mu_i z} + C_{-i}\boldsymbol{\Psi}_{-i}^{(k)}e^{-\mu_i z}], \quad (7)$$

where  $C_0$ ,  $C'_0$ ,  $C_i$  and  $C_{-i}$  are constants of linear combination, and

$$\boldsymbol{\Psi}_i^{(k)} = \begin{bmatrix} \psi_{i1}^{(k)} \\ \psi_{i2}^{(k)} \\ \psi_{i3}^{(k)} \\ \psi_{i4}^{(k)} \end{bmatrix} = \sum_{j=1}^2 A_{jk} \begin{bmatrix} f_1(\kappa_j, \mu_i, r) \\ f_2(\kappa_j, \mu_i, r) \\ f_3(\kappa_j, \mu_i, r) \\ f_4(\kappa_j, \mu_i, r) \end{bmatrix}_k + B_{jk} \begin{bmatrix} g_1(\kappa_j, \mu_i, r) \\ g_2(\kappa_j, \mu_i, r) \\ g_3(\kappa_j, \mu_i, r) \\ g_4(\kappa_j, \mu_i, r) \end{bmatrix}_k, \quad \boldsymbol{\Psi}_{-i}^{(k)} = \begin{bmatrix} \psi_{-i1}^{(k)} \\ \psi_{-i2}^{(k)} \\ \psi_{-i3}^{(k)} \\ \psi_{-i4}^{(k)} \end{bmatrix} = \sum_{j=1}^2 A_{jk} \begin{bmatrix} f_1(\kappa_j, -\mu_i, r) \\ f_2(\kappa_j, -\mu_i, r) \\ f_3(\kappa_j, -\mu_i, r) \\ f_4(\kappa_j, -\mu_i, r) \end{bmatrix}_k + B_{jk} \begin{bmatrix} g_1(\kappa_j, -\mu_i, r) \\ g_2(\kappa_j, -\mu_i, r) \\ g_3(\kappa_j, -\mu_i, r) \\ g_4(\kappa_j, -\mu_i, r) \end{bmatrix}_k.$$

## 5. Satisfaction of the end conditions and determining the unknown constants

The complete solution of  $k$ -th layer is required to satisfy the end conditions at  $z = 0$  and  $z = l$ . To this end, imposing end conditions on Eq. (7), we obtain

$$\mathbf{F}_{0k}(r) = C_0\boldsymbol{\Psi}_0^{(0k)} + C'_0\boldsymbol{\Psi}_0^{(1k)} + \sum_{i=1}^{\infty} [C_i\boldsymbol{\Psi}_i^{(k)} + C_{-i}\boldsymbol{\Psi}_{-i}^{(k)}] \quad (8)$$

$$\mathbf{F}_{lk}(r) = C_0\boldsymbol{\Psi}_0^{(0k)} + C'_0[\boldsymbol{\Psi}_0^{(1k)} + h\boldsymbol{\Psi}_0^{(0k)}] + \sum_{i=1}^{\infty} [C_i\boldsymbol{\Psi}_i^{(k)}e^{\mu_i l} + C_{-i}\boldsymbol{\Psi}_{-i}^{(k)}e^{-\mu_i l}] \quad (9)$$

where  $\mathbf{F}_{0k}(r) = [0 \quad 0 \quad r\sigma_{rz}(r, 0) \quad r\sigma_{zz}(r, 0)]_k^T$ ,  $\mathbf{F}_{lk}(r) = [u_r(r, l) \quad u_z(r, l) \quad 0 \quad rp(r)]_k^T$ , in which  $C_0$ ,  $C'_0$ ,  $C_i$  and  $C_{-i}$  are constants to be determined,  $\mathbf{F}_{0k}(r)$  contains the unknown tractions  $[\sigma_{rz}(r, 0)]_k$  and  $[\sigma_{zz}(r, 0)]_k$  at the fixed end,  $\mathbf{F}_{lk}(r)$  contains the unknown displacements  $[u_r(r, l)]_k$  and  $[u_z(r, l)]_k$  at the free end.

Multiplying both sides of Eq. (8) and (9) by  $\Psi_{-ik}^T \mathbf{J}$ ,  $\Psi_{ik}^T \mathbf{J}$ ,  $[\Psi_0^{(1k)}]^T \mathbf{J}$  and  $[\Psi_0^{(0k)}]^T \mathbf{J}$ , respectively, and integrating them over over  $(c_{k-1}, c_k)$  and then summing from 1-st layer to  $m$ -th layer, making use of the symplectic orthogonality [4], we obtain

$$C_i = \sum_{k=1}^m \int_{c_{k-1}}^{c_k} \Psi_{-ik}^T(r) \mathbf{J} \mathbf{F}_{0k}(r) dr, \quad C_{-i} = - \sum_{k=1}^m \int_{c_{k-1}}^{c_k} \Psi_{ik}^T(r) \mathbf{J} \mathbf{F}_{0k}(r) dr, \quad (10)$$

$$C_0 = \sum_{k=1}^m \int_{c_{k-1}}^{c_k} [\Psi_0^{(1k)}(r)]^T \mathbf{J} \mathbf{F}_{0k}(r) dr, \quad C'_0 = - \sum_{k=1}^m \int_{c_{k-1}}^{c_k} [\Psi_0^{(0k)}(r)]^T \mathbf{J} \mathbf{F}_{0k}(r) dr, \quad (11)$$

$$C_i = e^{-\mu_i l} \sum_{k=1}^m \int_{c_{k-1}}^{c_k} \Psi_{-ik}^T(r) \mathbf{J} \mathbf{F}_{lk}(r) dr, \quad C_{-i} = -e^{\mu_i l} \sum_{k=1}^m \int_{c_{k-1}}^{c_k} \Psi_{ik}^T(r) \mathbf{J} \mathbf{F}_{lk}(r) dr, \quad (12)$$

$$C_0 + C'_0 l = \sum_{k=1}^m \int_{c_{k-1}}^{c_k} [\Psi_0^{(1k)}(r)]^T \mathbf{J} \mathbf{F}_{lk}(r) dr, \quad C'_0 = - \sum_{k=1}^m \int_{c_{k-1}}^{c_k} [\Psi_0^{(0k)}(r)]^T \mathbf{J} \mathbf{F}_{lk}(r) dr. \quad (13)$$

Substituting Eqs. (10)-(11) in Eqs. (12)-(13) gives

$$\sum_{k=1}^m \int_{c_{k-1}}^{c_k} \Psi_{-ik}^T(r) \mathbf{J} [\mathbf{F}_{0k}(r) - e^{-\mu_i l} \mathbf{F}_{lk}(r)] dr = 0, \quad (14)$$

$$\sum_{k=1}^m \int_{c_{k-1}}^{c_k} \Psi_{ik}^T(r) \mathbf{J} [\mathbf{F}_{0k}(r) - e^{\mu_i l} \mathbf{F}_{lk}(r)] dr = 0, \quad (15)$$

$$\sum_{k=1}^m \int_{c_{k-1}}^{c_k} [\Psi_0^{(0k)}(r)]^T \mathbf{J} [\mathbf{F}_{0k}(r) - \mathbf{F}_{lk}(r)] dr = 0, \quad (16)$$

$$\sum_{k=1}^m \int_{c_{k-1}}^{c_k} \{ [\Psi_0^{(1k)}(r)]^T \mathbf{J} [\mathbf{F}_{0k}(r) - \mathbf{F}_{lk}(r)] - l [\Psi_0^{(0k)}(r)]^T \mathbf{J} \mathbf{F}_{0k}(r) \} dr = 0. \quad (17)$$

Taking  $n$  terms of the series for computation, we obtain a system of  $2n + 2$  algebraic equations ( $2n$  equations from Eq. (14) and Eq. (15) for  $i = 1, 2, \dots, n$ ; 2 equations from Eq. (16) and Eq. (17) for  $i = 0$ ) for the  $2n + 2$  unknowns:  $C_0$ ,  $C'_0$ ,  $C_i$  and  $C_{-i}$  ( $i = 1, 2, \dots, n$ ), which can be solved by using a standard method for simultaneous algebraic equations.

## Acknowledgements

The work is supported by the National Science Council of Taiwan, ROC through grants NSC99-2221-E006-067-MY3.

## References

- [1] J.Q. Tarn, A State Space Formalism for Anisotropic Elasticity, Part II: Cylindrical Anisotropy, International Journal of Solids and Structures. 39 (2002) 5157-5172.
- [2] J.Q. Tarn, W.D. Tseng, H.H. Chang, A Circular Elastic Cylinder Under Its Own Weight, International Journal of Solids and Structures. 46 (2009) 2886-2896.
- [3] F.B. Hildebrand, Advanced Calculus for Applications, 2nd Ed., Prentice-Hall, Englewood Cliffs, New Jersey, 1976.
- [4] W.X. Zhong, A New Systematic Methodology for Theory of Elasticity (in Chinese), Dalian University of Technology Press, Dalian, China, 1995.